MATHEMATICS METHODS FOR ENGINEERING

QUESTION 6:

Verify Stokes's theorem for the vector $F = zi + xj + yk$ and the surface S represented by the come $z = \sqrt{x^2 + y^2}$ where $0 \le z \le 4$.

Let us proof that $\int_R (\iint curl F).n\,dA = \oint_c F.dr$

$r = [\cos s, \sin s, 0]$ $\quad r' = [-\sin s, \cos s, 0]$ $\quad F_1 = zi$ $\quad F_2 = xj$

$F_3 = yk$

$F(r(s)) = [0, \cos s, \sin s]$

$$\oint_c F.dr = \int_0^{2\Pi} F(r(s).r'(s)\,ds$$

$$= \int_0^{2\Pi} [0 + \cos s \times \cos s + 0]\,ds$$

$$= \int_0^{2\Pi} \cos^2 s\,ds$$

$$= \int_0^{2\Pi} ds + \int_0^{2\Pi} \cos 2s\,ds$$

$$= s\Big|_0^{2\Pi} + 0$$

$$= 2\Pi$$

We now consider the surface integral, we have $F_1 = zi$ $F_2 = xj$ $F_3 = yk$ so can now calculate the $curl$.

$$curl F = \left(\frac{dy}{dy} - \frac{dx}{dz}\right) N_1 + \left(\frac{dz}{dz} - \frac{dy}{dx}\right) N_2 + \left(\frac{dx}{dx} - \frac{dz}{dy}\right) N_3$$

$$= curl(z, x, y)$$
$$= (1,1,1)$$

A normal vector of S is $N = grad(z - f(x,y)) = [2x, 2y, 2z]$. Hence

$(curl F) \bullet N = 2x + 2y + \frac{1}{16}$. Now $ndA = N dx dy$, using polar coordinates r, θ defined by $x = r\cos\theta, y = r\sin\theta$ and denoting the projection of S into the xy- plane by R, we thus obtain:

$$\iint_R curl F.ndA = \iint_R curl F.N dx dy$$

$$= \iint_R \left(x + y + \frac{1}{16}\right) dx dy$$

$$= \int_0^{2\Pi} \int_0^4 \left(2r(\cos\theta + \sin\theta) + \frac{1}{16}\right) r dr d\theta$$

$$= \int_0^{2\Pi} 32(\cos\theta + \sin\theta) + 1 d\theta$$

$$= 0 + \theta\Big|_0^{2\Pi}$$

$$= 2\Pi$$